

Scattering of twist fields from D-branes and orientifolds

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ABSTRACT: We compute the two-point function for Z_N orbifold twist fields on the disk and RP^2 . We apply this to a computation of the $\mathcal{O}(g_s)$ correction to the Kähler potential for (the symmetric combination of) blow-up modes in type I string theory on T^6/\mathbb{Z}_3 . This is related by supersymmetry to the one-loop field dependent correction to the Fayet-Iliopoulos D-term for the anomalous U(1) factor. We find this correction to be non-vanishing away from the orbifold point.

KEYWORDS: Conformal Field Models in String Theory, Compactification and String Models.

Contents

1. Introduction	2
2. Bosonic correlation functions	3
2.1 Boundary conditions	3
2.2 Correlation functions and stress tensor insertions	4
2.3 The quantum piece	5
2.3.1 The disk ($\epsilon_2 = 1$)	7
2.3.2 The projective plane ($\epsilon_2 = -1$)	9
2.4 The classical piece	10
2.4.1 Monodromy and boundary conditions — choosing contours	11
2.4.2 Monodromies and boundary conditions — the coset lattice	12
2.4.3 The full instanton sum	12
2.4.4 Relation to Dixon et. al.	13
3. Fermionic correlation functions	13
4. Twisted sector kinetic terms to $\mathcal{O}(g_s)$ in type I on T^6/\mathbb{Z}_3	15
4.1 The T^6/\mathbb{Z}_3 orbifold and the 4d effective action	15
4.2 The closed string vertex operator	17
4.3 Computing the two-point amplitudes	17
4.3.1 The ghost and spacetime parts of the amplitudes	19
4.3.2 Twist field correlators and the coset lattice	19
4.3.3 The full amplitude	19
4.4 Connecting the pieces	20
4.4.1 The amplitudes near $x = -1$ — continuity constraint	21
4.4.2 The amplitude near $x = 0, \pm\infty$ — tadpole cancellation constraints	21
4.4.3 The amplitude near $x = 1$ and open string dynamics	22
4.5 The full amplitude on T^3/\mathbb{Z}_3	23
5. Conclusions	25
A. Definition of \tilde{F}	25
B. Some useful hypergeometric integrals	26

1. Introduction

In this work we will consider the scattering of orbifold blow-up modes off of D-branes and orientifold planes. Our motivation is the study of the effective action of 4d models with open strings and low-energy $\mathcal{N} = 1$ spacetime supersymmetry. Orbifold blow-up modes are typically moduli or light scalars in the problem, and control some of the coupling constants in the open string sector.

More precisely, we will compute the two-point function of orbifold twist fields on the disk and RP^2 . Among other things, this allows us to compute the $\mathcal{O}(g_s)$ correction to the quadratic part of the Kähler potential for the blow-up modes. To the best of our knowledge such computations have not been done for twisted sector moduli — most of the focus has been on corrections to the Kähler potential for the untwisted moduli, as in [1, 2].¹ For twisted moduli, this term is important for computing the Fayet-Iliopoulos D-terms in these models, as we will discuss in section 4.

The difficulty in this calculation stems from the correlators for the bosonic twist fields. For the CFT of a free boson or free fermion, one may use the "doubling trick". This trick constructs the disk or real projective plane from an identification on the complex plane, and uses this identification to relate the antiholomorphic part of the operators on the disk or real projective plane to a holomorphic part on the complex plane. Thus all correlators can be transformed to holomorphic correlators on the sphere. However, this trick requires that one can factorize the operators into holomorphic and antiholomorphic pieces. For bosonic twist fields, this is not possible, as can be seen from the calculation of the four-point function of twist operators on the sphere [4].

We will compute the two-point function of twist fields on the disk and on RP^2 for a variety of boundary conditions, using the "stress tensor method" in [4]. Our calculation will amount to performing the doubling trick on the additional insertions in [4] and mapping all of the monodromy integrals to holomorphic integrals on the sphere. The nonfactorizability of the bosonic twist fields will arise from the non-locality of the monodromy conditions on the sphere. The result is very close to the calculation of the 4-point function on the sphere.

We then apply this to a calculation of the $\mathcal{O}(g_s)$ correction to the spacetime kinetic term for twisted sector moduli in the type I T^6/\mathbb{Z}_3 orbifold [5]. After canceling all of the tadpoles and unphysical divergences, the different contributions to the amplitude (coming from the disk and the projective plane) join at the boundary of their moduli spaces into a single integral over the modular parameter of four points on the sphere, leading to a finite non-zero answer. We are unable to evaluate that integral analytically, or even numerically for finite volume of the torus; however, we have done the integral numerically in the limit of large volume of the orbifold, and found it to be non-zero.²

The outline of this paper is as follows. In section 2 we will compute the two-point-function of bosonic twist fields on the disk and real projective plane. In section 3 we will

¹See [3] for a generalization of [1].

²Ref. [6], which appeared as this paper was being completed, has also argued that the one-loop correction to the FI D-term is nonvanishing away from the orientifold point in type IIA models on T^6 with intersecting 6-branes.

compute the correlation function of fermionic twist fields. In section 4 we will combine these to calculate the two-point function of the diagonal sum of the twisted sector moduli of type I string theory compactified on T^6/\mathbb{Z}_3 . We study the divergence structure of these amplitudes, discuss their spacetime interpretation, and extract the one-loop correction to the Kähler potential. In section 5 we will conclude with some comments on the space-time physics of this calculation. The appendices collect some facts about hypergeometric functions.

2. Bosonic correlation functions

In this section we wish to compute the two-point function of bosonic \mathbb{Z}_N twist fields on worldsheets with disk and RP^2 topology, for the case of T^2/\mathbb{Z}_N and \mathbb{C}/\mathbb{Z}_N . The T^2 and \mathbb{C} are described by a complex coordinate X and \mathbb{Z}_N is generated by the action $X \rightarrow X e^{2\pi i/N}$.

We will use the stress tensor method for this calculation, as developed in [4]. The essence of this method is as follows. In order to compute the two-point function of two twist fields $\sigma_1(z_1, \bar{z}_1), \sigma_2(z_2, \bar{z}_2)$, we compute this two-point function with the insertion $\partial X(z) \partial \bar{X}(w)$. The monodromy conditions on $\partial X, \partial \bar{X}$ determine this correlator. The leading nonsingular part of this composite operator as $z \rightarrow w$ is the stress tensor, and we can then integrate the conformal Ward identities to extract the original correlator.

Although we cannot apply the doubling trick directly to the twist fields, we *can* apply it to the $\partial X(z) \partial \bar{X}(w)$ insertion. This application, and the modification of the monodromy conditions to the disk and RP^2 topology, are the essential innovations of this section.

2.1 Boundary conditions

Boundary conditions on X can be imposed by considering X as a function on the complex plane, and then projecting onto configurations invariant under the \mathbb{Z}_2 action

$$z \rightarrow \epsilon_2 / \bar{z} . \tag{2.1}$$

The fundamental domain of this map is the unit disk $|z| \leq 1$. It has the topology of a disk for $\epsilon_2 = 1$ and of RP^2 for $\epsilon_2 = -1$. The action of this symmetry on X is:

$$X(z, \bar{z}) = \epsilon_1 X(\epsilon_2 / \bar{z}, \epsilon_2 / z) . \tag{2.2}$$

Here $\epsilon_1 = 1$ (-1) for the coordinates parallel (perpendicular) to the brane/orientifold plane, and we choose the brane/orientifold to be located at $X = 0$ in the directions perpendicular to it.

Using (2.2), the holomorphic and anti-holomorphic fields — ∂X and $\bar{\partial} X$, respectively — in the fundamental region can be combined into a single holomorphic function on the sphere by defining:

$$\partial X(z) = \epsilon_1 \bar{\partial} X(\bar{u}) \left(\frac{\partial \bar{u}}{\partial z} \right)^{(\Delta=1)} \Big|_{\bar{u}=\epsilon_2/z} = -\epsilon_1 \epsilon_2 \bar{\partial} X(\epsilon_2/z) / z^2 \tag{2.3}$$

for $|z| > 1$, or equivalently

$$z\partial X(z) = -\epsilon_1 \bar{u} \bar{\partial} X(\bar{u}) \Big|_{\bar{u}=\epsilon_2/z} . \quad (2.4)$$

Note that the \mathbb{Z}_2 (2.1) acts on ∂X with the Jacobian factor expected of an operator with dimension $\Delta = 1$.

We are interested in the correlators

$$C = \langle \partial X(z) \partial \bar{X}(w) \sigma_1(z_1, \bar{z}_1) \sigma_2(z_2, \bar{z}_2) \rangle , \quad (2.5)$$

where $\sigma_{1,2}$ are bosonic twist fields. As functions of z, w these satisfy additional boundary conditions as $z, w \rightarrow z_{1,2}$. The correlators will be further constrained by the interaction of these boundary conditions with (2.3).

More precisely, consider a bosonic twist operator σ_+ that twists the field $X(z, \bar{z})$ by $e^{2\pi i k/N}$ at (w, \bar{w}) . The boson X has the following OPEs with σ_+ [4]:

$$\begin{aligned} \partial X(z) \sigma_+(w, \bar{w}) &\sim (z-w)^{-(1-k/N)} \tau_+(w, \bar{w}) \\ \bar{\partial} X(\bar{z}) \sigma_+(w, \bar{w}) &\sim (\bar{z}-\bar{w})^{-k/N} \tilde{\tau}'_+(w, \bar{w}) \\ \partial \bar{X}(z) \sigma_+(w, \bar{w}) &\sim (z-w)^{-k/N} \tau'_+(w, \bar{w}) \\ \bar{\partial} \bar{X}(z) \sigma_+(w, \bar{w}) &\sim (\bar{z}-\bar{w})^{-(1-k/N)} \tilde{\tau}_+(w, \bar{w}) . \end{aligned} \quad (2.6)$$

Here $\tau, \tilde{\tau}, \tau', \tilde{\tau}'$ are known as excited twist fields. We also define the "conjugate twist field" σ_- as having the OPEs (2.6) with $k/N \leftrightarrow 1-k/N$, and $\tau_+ \leftrightarrow \tau_-$, etc.

2.2 Correlation functions and stress tensor insertions

In section 4, we will find that we need to compute the following two-point functions:

$$H_{-\pm}^{\epsilon_1, \epsilon_2} = \langle \sigma_-(z_1, \bar{z}_1) \sigma_{\pm}(z_2, \bar{z}_2) \rangle . \quad (2.7)$$

The quantum symmetry that would set $H_{--} = 0$ on the sphere is broken on the disk and on RP^2 . We will give a spacetime argument in section 4, but for now we note that we find no reason for it to vanish from a two-dimensional point of view, either.

The first step in computing H will be to compute the following connected correlators:

$$\begin{aligned} g_{-\pm}^{\epsilon_1, \epsilon_2}(z) &= \frac{\langle -\frac{1}{2} \partial X(z) \partial \bar{X}(z) \sigma_-(z_1, \bar{z}_1) \sigma_{\pm}(z_2, \bar{z}_2) \rangle}{\langle \sigma_-(z_1, \bar{z}_1) \sigma_{\pm}(z_2, \bar{z}_2) \rangle} \\ h_{-\pm}^{\epsilon_1, \epsilon_2}(z, \bar{z}) &= \frac{\langle -\frac{1}{2} \bar{\partial} X(\bar{z}) \partial \bar{X}(z) \sigma_-(z_1, \bar{z}_1) \sigma_{\pm}(z_2, \bar{z}_2) \rangle}{\langle \sigma_-(z_1, \bar{z}_1) \sigma_{\pm}(z_2, \bar{z}_2) \rangle} . \end{aligned} \quad (2.8)$$

These are determined by imposing the boundary conditions and monodromy conditions on the z -dependence of these correlators, as in [4].

The monodromy conditions arise from the fact that as one takes X around noncontractible contours on the worldsheet, which surround the same amount of Z_N twist as anti-twist, X should return to itself up to an element of the toroidal orbifold group (which

define the torus.) figures 1-3 show some of the contours which contribute to the monodromy conditions on the disk for H_{-+} , for H_{--} with $\epsilon_1 = 1$, and for H_{--} with $\epsilon_1 = -1$, respectively. Figure 4 shows such contours for RP^2 , for H_{-+} . The contours inside the disk are noncontractible contours, with the ends at $|z| = 1$ identified via the action (2.1). The full closed loops are the images of these contours under (2.1).

We will find that it is enough to impose a single monodromy condition for each case. The contours that contribute nontrivially to the monodromy conditions for $\epsilon_1 = \pm 1$, will be trivially satisfied or will give equivalent monodromy conditions for $\epsilon_1 = \mp 1$. This is unlike the calculation in [4] of the four-point function on the sphere, for which there are two independent monodromy conditions that must be enforced.

The monodromy conditions on X inside correlation functions $H_{-\pm}$ are:

$$\int_C (dz\partial X + d\bar{z}\bar{\partial}X) = v, \tag{2.9}$$

where v is an element of some coset of the Narain lattice Λ [4]; we will discuss this further in section 2.4. In our case there is a further simplification — the boundary conditions (2.2) allow one to take the contour integrals above, and write them as *holomorphic* contours in the full complex plane. These contours are also shown in figures 1-4.

In performing the path integral to compute H , we sum over the different possible monodromies, as we will discuss in section 2.4. This sum can be simplified as follows. For a given monodromy (2.9), we split X into two pieces $X = X_{cl} + X_q$. The first is the "classical" piece, it satisfies the classical equations of motion $\partial\bar{\partial}X_{cl} = 0$ and the conditions (2.9) by itself. The second piece X_q is the "quantum" piece containing the fluctuations about the classical solution, and solves (2.9) with $v = 0$. The full correlator can be written as

$$\begin{aligned} H_{-\pm}^{\epsilon_1, \epsilon_2} &= H_{q; -\pm}^{\epsilon_1, \epsilon_2} H_{cl; -\pm}^{\epsilon_1, \epsilon_2} \\ H_{cl; -\pm}^{\epsilon_1, \epsilon_2} &= \sum_v e^{-S_{cl, -\pm}^{\epsilon_1, \epsilon_2}[v]}, \end{aligned} \tag{2.10}$$

where in H_{cl} the sum is over the elements in the corresponding coset lattice (2.9).

The connected correlation functions $g_{\mp,+}$ are independent of the classical piece. They are determined by the boundary conditions, together with the monodromy conditions (2.9) for $v = 0$. Once g is determined, one can use the conformal Ward identities to compute H_q . We will give an outline of and results of this computation in section 2.3. The sum over classical solutions will be described in section 2.4.

2.3 The quantum piece

We will compute $H_{q; -\pm}^{\epsilon_1, \epsilon_2}$ case by case. Before presenting the answers we will sketch the basic strategy.

To construct H_q we need first to compute $g_{-\pm}$. Because g is a connected correlator, any dependence on the explicit factor of ∂X_{cl} vanishes, and it can be taken as the connected correlator of $\partial X_q(z)\partial\bar{X}_q(w)$.

The asymptotic conditions as z approaches any of the other insertions in (2.8), together with the condition (2.3), determines $g_{\mp,+}(z, w)$ up to some function $A_{\pm}(z_1, \bar{z}_1, z_2, \bar{z}_2)$:

$$g_{-+}(z, w) = \omega_k(z)\omega_{N-k}(w) \left\{ \frac{k}{N} \frac{(z - z_1)(z - 1/\bar{z}_2)(w - z_2)(w - 1/\bar{z}_1)}{(z - w)^2} + \left(1 - \frac{k}{N}\right) \frac{(z - z_2)(z - 1/\bar{z}_1)(w - z_1)(w - 1/\bar{z}_2)}{(z - w)^2} + A_+ \right\}, \quad (2.11)$$

where

$$\omega_k(z) \equiv [(z - z_1)(z - \epsilon_2/\bar{z}_2)]^{-k/N} [(z - z_2)(z - \epsilon_2/\bar{z}_1)]^{-(1-k/N)}. \quad (2.12)$$

Similarly, g_{--} is obtained from g_{-+} by exchanging $z_2 \leftrightarrow \epsilon_2/\bar{z}_2$, and $A_+ \leftrightarrow A_-$.

A is determined by the monodromy condition for X_{qu} , which is (2.9) with $v = 0$ for an appropriate choice of contour C . We will call these contours closed if their image on the branched cover of the sphere, as given by (2.1), is closed.

Consider the space of all closed contours in $|z| \leq 1$, up to homotopy. In our case this space is one-dimensional, with a single contour \mathcal{L}_0 as a basis element. (In the four-point function on the sphere, the space is two-dimensional [4]). By using the map (2.1), the monodromy integral (2.9) over the contour \mathcal{L}_0 in the fundamental region can be mapped to an integral over a single closed loop γ on the sphere (the covering space), times a numerical factor $1/k$, where $k = (3 - \epsilon_1\epsilon_2)/2$ (this will be unimportant for the monodromy condition for X_q , but important for X_{cl}). The map (2.1) maps a twist operator to an anti-twist operator at the image point: as a result, the total amount of twist encircled by the loop γ on the sphere is zero (and one may think of the monodromy condition for X_{qu} as a monodromy condition on the sphere.) The fundamental closed loop on the sphere γ is drawn for each case in figures 1-4. In these figures, the original contour C inside the fundamental region $|z| \leq 1$ is black, while the remainder $\gamma - C$ outside the fundamental region is blue. The monodromy condition for X_{qu} can be written as:

$$\Delta_{\mathcal{L}_0} X_{qu} = \frac{1}{k} \oint_{\gamma} dz \partial X_{qu} + \frac{1}{k} \oint_{\gamma} d\bar{z} \bar{\partial} X_{qu} = 0. \quad (2.13)$$

Next, we can turn the left hand side of (2.13) into a single holomorphic integral. Using (2.1), the condition (2.9) for X_q becomes:

$$\oint_{\gamma} dz \partial X_{qu} + \epsilon_1 \oint_{\bar{\gamma}} dz \partial X_{qu} = 0, \quad (2.14)$$

where $\bar{\gamma}$ (also shown in figures 1-4) is the image of γ under (2.1). Finally, we can apply (2.14) to $g_{-\pm}$ to find that:

$$\oint_{\gamma} dz g_{-\pm} + \epsilon_1 \oint_{\bar{\gamma}} dz g_{-\pm} = 0. \quad (2.15)$$

This equation can be solved for A_{\pm} , and so determines g_{\pm} .

Given $g_{-\pm}$, we then use the conformal Ward identities to extract and solve a differential equation for $H_{q;-\pm}$. First, we can use the $SL(2, \mathbb{R})$ symmetry of correlators on the disk

and RP^2 to set $(z_1, z_2) = (0, y)$, where $y \in [0, 1]$. Taking $z \rightarrow w$ we have

$$\begin{aligned} \frac{\langle T(z)\sigma_-(0,0)\sigma_\pm(y,\bar{y}) \rangle}{\langle \sigma_-(0,0)\sigma_\pm(y,\bar{y}) \rangle} \Big|_{\bar{y}=y} &= \lim_{z \rightarrow w} \left[g_\pm(z, w) - \frac{1}{(z-w)^2} \right] \\ &= \frac{1}{2} \frac{k}{N} \left(1 - \frac{k}{N} \right) \left(\frac{1}{z} \pm \frac{1}{z - \epsilon_2/y} \mp \frac{1}{z-y} \right)^2 + \frac{\tilde{A}}{z(z - \epsilon_2/y)(z-y)}, \end{aligned} \quad (2.16)$$

where

$$\tilde{A}_\pm(y) \equiv \lim_{z_1 \rightarrow 0} -\epsilon_2 \bar{z}_1 A_\pm(z_1, y, \epsilon_2/y, \epsilon_2/\bar{z}_1). \quad (2.17)$$

As $z \rightarrow y$ we have the operator product expansion

$$T(z)\sigma_\pm(y) \sim \frac{h_\sigma \sigma_\pm(y)}{(z-y)^2} + \frac{\partial \sigma_\pm(y)}{(z-y)} + \dots \quad (2.18)$$

Integrating along a small contour around y and noting that H_q is a function of $y\bar{y}$ only (we have used the $SL(2, \mathbb{R})$ symmetry to set $y = \bar{y}$), leads to

$$\begin{aligned} \left[\partial_y \ln H_{q;-\pm}^{\epsilon_1, \epsilon_2}(y\bar{y}) \right] \Big|_{y=\bar{y}} &= \frac{1}{2} \partial_y \ln H_{q;-\pm}^{\epsilon_1, \epsilon_2}(y^2) \\ &= \mp \frac{k}{N} \left(1 - \frac{k}{N} \right) \left(\frac{1}{y} \pm \frac{1}{y - \epsilon_2/y} \right) + \frac{\tilde{A}_\pm}{y(y - \epsilon_2/y)}. \end{aligned} \quad (2.19)$$

After solving for $H_{q;-pm}^{\epsilon_1, \epsilon_2}(y^2)$, we will rewrite it as a function of the cross-ratio on the sphere

$$x \equiv \left[\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \right]^{\pm 1} \Big|_{(z_1, z_2, z_3, z_4) = (0, y, \epsilon_2/y, \infty)} = \epsilon_2 y^{\pm 2}, \quad (2.20)$$

where the sign of the exponent corresponds to the sign of the twist field inserted at (z_2, \bar{z}_2) . Let us now consider each case in turn.

2.3.1 The disk ($\epsilon_2 = 1$)

For each value of ϵ_1 on the disk, there is a single nontrivial closed contour allowed by the boundary conditions. For $\epsilon_1 = 1$, these contours and their images under (2.1) are shown in figures 1a (for g_{-+}) and 2 (for g_{--}). For $\epsilon_1 = -1$ the contours are shown in figure 1b (for g_{-+}) and 3 (for g_{--}). We can insert these contours into (2.15) to find the equation for A_\pm .

To see that only one of these two loops on the sphere contribute for each correlator, note that by deforming the loops around infinity in the complex plane,³ one can show that in figures 1 and 3, $\bar{\gamma}_1 = \epsilon_1 \gamma_1$, while in figure 2, $\bar{\gamma}_2 - \gamma_2$ is proportional to γ_1 in figure 3 plus the image of the same contour on the second sheet of the cut plane. The result is that when $\epsilon_1 = -1$ (+1), the monodromy integrals (2.9) for the contours in Fig 1a and 2 (1b and 3) on the disk do not impose new conditions. Therefore, we can simplify the monodromy integrals to

$$\oint_{\gamma_i} dz g = 0, \quad (2.21)$$

³Note that there is no simple pole at infinity.

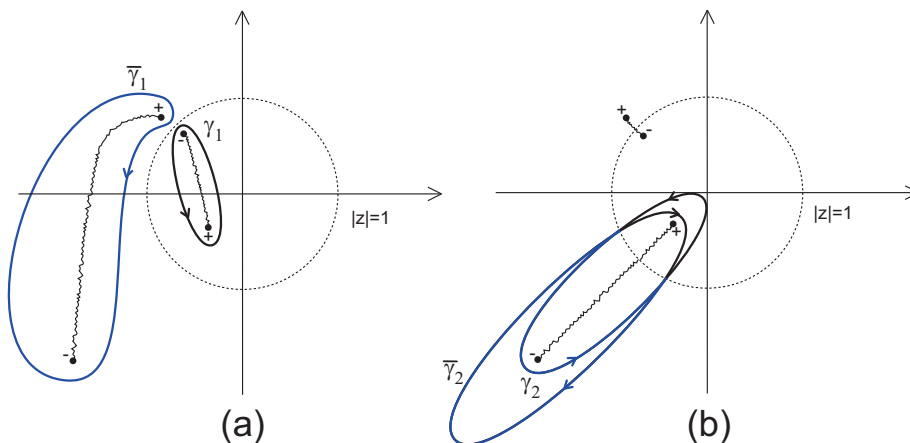


Figure 1: Noncontractible contours on the disk for H_{-+} . The interior of the dashed circle is the fundamental domain of eq. (2.1). The black contours inside the disk are noncontractible due to the boundary conditions; figures (a) and (b) shows the noncontractible loops which contribute to the monodromy conditions for $\epsilon_1 = 1, -1$ respectively. The blue contours are the images of the black contours under (2.1).

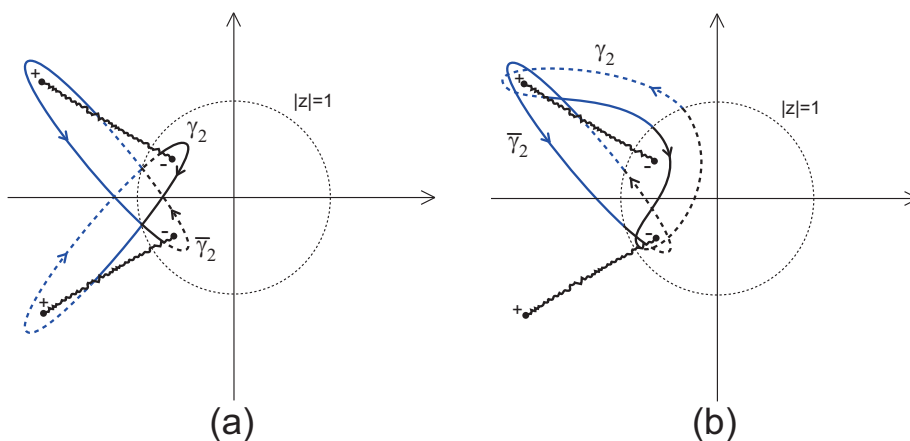


Figure 2: Noncontractible contours on the disk for H_{--} , which contribute to the monodromy conditions when $\epsilon_1 = 1$. Figures (a) and (b) are topologically equivalent contours.

where for $g_{-\pm}, i = 1(2)$ for $\epsilon_1 = \pm 1(\mp 1)$.

We perform the contour integrals with the sole purpose of finding A_{\pm} . Since A is a function of (z_i, \bar{z}_i) only, we can set w in $g(z, w)$ to be what we like. Therefore, we divide the monodromy condition by $z_1^{-k/N} \omega_{N-k}$ and send $w \rightarrow \infty$. After performing the resulting integrals (with the help of the identities in appendix B) we find that:

$$\begin{aligned}
 H_{q; -+}^{+,+}(y) &= \text{const}[y^2(1-y^2)]^{-k/N(1-k/N)} \tilde{F}(y^2)^{-1} \\
 &= \text{const}[x(1-x)]^{-k/N(1-k/N)} \tilde{F}(x)^{-1} \\
 H_{q; --}^{+,+}(y) &= \text{const}(1-y^{-2})^{-k/N(1-k/N)} \tilde{F}(y^{-2})^{-1} \\
 &= \text{const}(1-x)^{-k/N(1-k/N)} \tilde{F}(x)^{-1}
 \end{aligned}$$

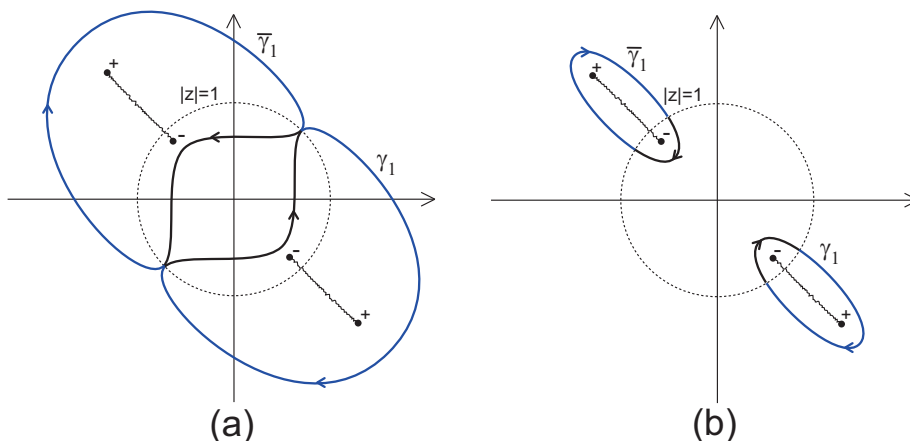


Figure 3: Noncontractible contours on the disk for H_{--} , which contribute to the monodromy conditions when $\epsilon_1 = -1$. Figures (a) and (b) are topologically equivalent contours.

$$\begin{aligned}
 H_{q; -+}^{-,+}(y) &= \text{const}[y^2(1-y^2)]^{-k/N(1-k/N)} F(1-y^2)^{-1} \\
 &= \text{const}[x(1-x)]^{-k/N(1-k/N)} F(1-x)^{-1} \\
 H_{q; --}^{-,+}(y) &= \text{const}(1-y^{-2})^{-k/N(1-k/N)} F(1-y^{-2})^{-1} \\
 &= \text{const}(1-x)^{-k/N(1-k/N)} F(1-x)^{-1}, \tag{2.22}
 \end{aligned}$$

where x is defined in (2.20), and $\tilde{F}(x) = \tilde{F}(1-k/N, k/N, 1; x)$ is defined in appendix A.

2.3.2 The projective plane ($\epsilon_2 = -1$)

We proceed as in the case of the disk. As before, for $\epsilon_1 = 1$ we find that the sole relevant monodromy condition is:

$$\oint_{\gamma_1} dzg_{-+} = 0, \tag{2.23}$$

where γ_1 is drawn in figure 4a for g_{-+} , and we have found that it is equal to $\bar{\gamma}_1$ by deforming the loop around infinity in the complex plane. Similarly, by deforming the loop $\bar{\gamma}_2$ in Fig 4b (see Fig 5), we find that:

$$\oint_{\bar{\gamma}_2} dzg = - \oint_{\gamma_2} dzg + (1 - e^{2\pi ik/N}) \oint_{\gamma_1} dzg, \tag{2.24}$$

so that for $\epsilon_1 = 1$, the monodromy condition for the loop γ_2 reads:

$$\oint_{\gamma_2} dzg + \oint_{\bar{\gamma}_2} dzg = (1 - e^{2\pi ik/N}) \oint_{\gamma_1} dzg = 0 \tag{2.25}$$

and is therefore equivalent to the monodromy condition obtained from the loop γ_1 .

For $\epsilon_1 = -1$, the monodromy condition obtained from the loop γ_1 is trivially satisfied (as $\gamma_1 = \bar{\gamma}_1$), whereas the monodromy condition obtained from the loop γ_2 is non-trivial.⁴

⁴For H_{--}^{+-} one may equivalently consider the loop in Fig 3a.

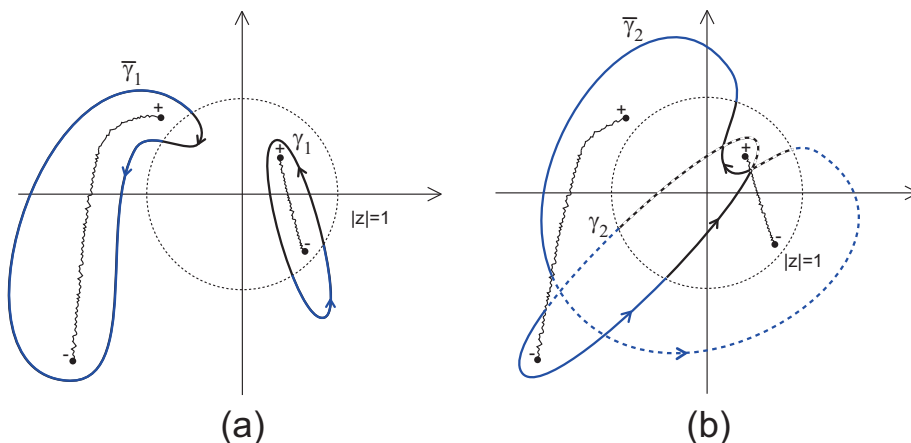


Figure 4: Noncontractible loops for g_{-+} on RP^2 . Figures (a) and (b) are for $\epsilon_1 = +1$ and -1 , respectively.

As before, we divide the monodromy condition ((2.23) or (2.25)) by $z_1^{-k/N} \omega_{N-k}(w)$, send $w \rightarrow \infty$ and use the $SL(2, \mathbb{R})$ symmetry to set $(z_1, z_2) = (0, y)$, where $y \in [0, 1]$. The results of the calculation are:

$$\begin{aligned}
 H_{q; -+}^{+, -}(y) &= \text{const}[y^2(1+y^2)]^{-k/N(1-k/N)} F(-y^2)^{-1} \\
 &= \text{const}[x(x-1)]^{-k/N(1-k/N)} F(x)^{-1} \\
 H_{q; --}^{+, -}(y) &= \text{const}(1+y^{-2})^{-k/N(1-k/N)} F(-y^{-2})^{-1} \\
 &= \text{const}(x-1)^{-k/N(1-k/N)} F(x)^{-1} \\
 H_{-+; qu}^{-, -}(y) &= \text{const}[y^2(1+y^2)]^{-k/N(1-k/N)} \tilde{F}(1+y^2)^{-1} \\
 &= \text{const}[x(x-1)]^{-k/N(1-k/N)} \tilde{F}(1-x)^{-1} \\
 H_{q; --}^{-, -}(y) &= \text{const}(1+y^{-2})^{-k/N(1-k/N)} \tilde{F}(1+y^{-2})^{-1} \\
 &= \text{const}(x-1)^{-k/N(1-k/N)} \tilde{F}(1-x)^{-1} .
 \end{aligned} \tag{2.26}$$

2.4 The classical piece

If we are studying an orbifold of a torus, then there are also classical contribution to the correlators, coming from worldsheet instantons. These are solutions to $\partial\bar{\partial}X = 0$ which satisfy the monodromy conditions (2.9), for the contour and values of v appropriate to the boundary conditions.

A general element of the combined orbifold group acts as

$$X - X_0 \rightarrow e^{2\pi ik/N}(X - X_0) + v, \tag{2.27}$$

where $v \in \Lambda$ is a point in the Narain lattice and X_0 is a fixed point. In this subsection we will set $X_0 = 0$ and assume that the two twist operators belong to the same fixed point. (The generalization to other fixed points, and to correlators of twist operators belonging to different fixed points, are easily obtained from our results.) Twist fields are built from local worldsheet insertions about which X has this monodromy. Following [4], we will label such insertions by (θ^k, v) , where θ is the generator of Z_N .

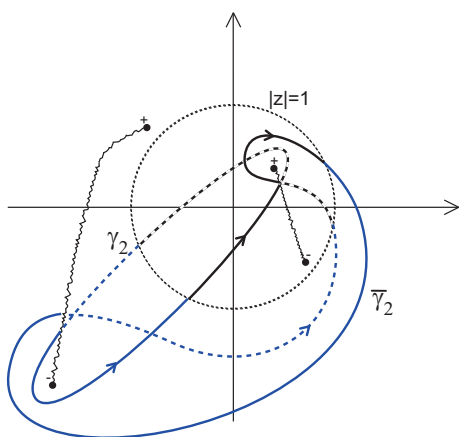


Figure 5: The same loops as in figure 4.b after the loop $\tilde{\gamma}_2$ has been deformed around infinity.

In order to construct a twist operator that is invariant under the orbifold group, we have to sum over all of its images under conjugation by elements of the orbifold group. Starting from some element (θ^j, v_0) and conjugating it by (θ^k, u) , we get $(\theta^j, \theta^k v_0 + (1 - \theta)u)$ [4]. Therefore, in the orbifold theory, a twist operator is labeled by a *point group* element θ^j and a *coset lattice* vector

$$\{\theta^k v_0 + (1 - \theta)u, k \in \mathbb{Z}, u \in \Lambda\} . \quad (2.28)$$

As we encircle $|z| = 1$, the field X transforms in a way dictated by the twist operators inserted inside. Since the combined orbifold group is non-commutative, that transformation depends on their ordering. Here we choose a radial ordering centered at the origin. Note that before summing over the coset lattice (and, by that, obtaining a good operator in the orbifold theory), radial orderings centered at different points (corresponding to different conformal frames) are not equivalent.

In addition to the orbifold projection onto states invariant under (2.27), the models we will consider also contain orientifolds, so we must perform an orientifold projection. An orientifold may wrap each of the T^2 factors parametrized by X ($\epsilon_1 = 1$) or be localized at one of the fixed points X_0 ($\epsilon_1 = -1$). A wrapped orientifold acts by Ω and a localized orientifold acts by $R\Omega$, where Ω is the worldsheet parity and R is the reflection

$$R : \quad X - X_0 \longrightarrow -(X - X_0) . \quad (2.29)$$

In the case of \mathbb{Z}_3 orbifolds of T^2 , the fixed point X_0 is invariant where the other two fixed points are identified. Again, here we set $X_0 = 0$.

2.4.1 Monodromy and boundary conditions — choosing contours

When computing the quantum piece in section 2.3, we learned that for $g_{-\pm}$ and $\epsilon_1 = \pm 1$, the non-trivial monodromy condition is obtained from the loops labeled γ_1 in figures 1-4, while for $g_{-\pm}$ and $\epsilon_1 = \mp 1$ it is obtained from γ_2 . One difference between the quantum and the classical cases is that for some correlators the two loops (γ_1 and γ_2) give equivalent monodromy conditions. For the classical piece however, one of the two conditions is always trivially satisfied.

To see this explicitly, note that for g_{-+} with $\{\epsilon_1 = 1, \epsilon_2 = -1\}$, and for g_{--} with $\{\epsilon_1 = -1, \epsilon_2 = 1\}$ the monodromy condition obtained by translating the field X around the loop γ_2 equals the monodromy condition obtained by translating the field X around the loop γ_1 , minus the monodromy condition obtained by translating the field θX around the loop γ_1 . Since the total change in the value of X_q along the closed loops was zero, the two loops γ_1 and γ_2 gave equivalent monodromy conditions, related by a non-zero number $(1 - \theta)$ (2.25). However, the total change in the value of X_{cl} along the closed loops is some

coset lattice vector. That coset lattice vector is dictated by the twist operators enclosed by the loop *and* their radial ordering.⁵ Let the twist operators encircled by γ_1 be (θ^i, v) at y and (θ^{-i}, u) at the origin. As we translate X_{cl} around γ_1 , it shifts by $v + \theta^i u$. The translation of X_{cl} around γ_2 is equivalent to the translation of X_{cl} around γ_1 , minus the translation of $\theta^i \tilde{X}_{cl}$ around γ_1 , where \tilde{X}_{cl} is the classical solution with the ordering of the two twists reversed. Now since $\theta^i(u + \theta^{-i}v) = v + \theta^i u$, the two shifts (of X_{cl} and $\theta^i \tilde{X}_{cl}$) cancel. Therefore, in these specific cases, the monodromy condition obtained by translating the field X_{cl} along the loop γ_2 is trivially satisfied with $v = 0$. Similarly for other boundary conditions and twist insertions, the monodromy condition for one contour (γ_1 or γ_2) is always trivially satisfied with $v = 0$. Therefore, the classical solutions are labeled by a single vector v ; one then performs the sum in (2.10) over the elements v of the appropriate coset lattice of the Narain lattice.

2.4.2 Monodromies and boundary conditions — the coset lattice

For the disk or RP^2 , the classical solution has to satisfy (2.1) at $|z| = 1$. This potentially affects the coset lattice which defines a good twist field in the conformal field theory.

For $\epsilon_1 = 1$, (2.1) is automatically satisfied. There is no constraint on the possible twist insertions inside the fundamental domain $|z| \leq 1$ of (2.1).

For $\epsilon_1 = -1$, the twist field insertions are restricted by the demand that there is no non-trivial monodromy on the boundary. That can be seen by noting that after moving the loop through the boundary, it returns with the opposite orientation, so that the corresponding monodromy must vanish.

Now, suppose that for $\epsilon_1 = -1$, we insert two twist fields, labeled by (θ^i, v) at the origin and (θ^j, u) at $0 < y < 1$. Consistency with the conditions at $|z| = 1$ requires that $u = -\theta^j v$.^{6,7} Therefore, while for $\epsilon_1 = 1$ we have two independent sums over u and v in the coset lattice for each of the two twist operators, for $\epsilon_1 = -1$ we have only one sum over v , with u constrained to be $u = -\theta^j v$.

In the cases $\epsilon_1 = -1$, the nontrivial monodromy conditions come from contours surrounding a twist field and its image under the map (2.2). The image of (θ, v) is $(\theta^{-1}, -\epsilon_1 \theta^{-1} v)$. Therefore, the monodromy integral for a loop that encircles a twist operator (θ, v) and its image $(\theta^{-1}, -\epsilon_1 \theta^{-1} v)$ on the sphere, is equal to $(\theta^{-1} - \epsilon_1 \theta^{-1})v$. Such a loop will be non-trivial only for $\epsilon_1 = -1$. In that case the coset lattice is rescaled by 2 — in other words, for $\epsilon_1 = -1$, we sum over even multiples of the vectors v which contribute for $\epsilon_1 = 1$.

2.4.3 The full instanton sum

By an explicit calculation we find that

$$H_{cl; -\pm}^{\epsilon_1, \epsilon_2} = \sum_v \exp \left(-\frac{\pi R^2 (5 - 3\epsilon_1)(5 - 3\epsilon_1 \epsilon_2)}{16\alpha' \sin(\pi k/N)} \left[\frac{\tilde{F}(1-x)}{\tilde{F}(x)} \right]^{\epsilon_1} |v|^2 \right), \quad (2.30)$$

⁵Since the point group is commutative, for X_g the ordering of the group elements was not an issue.

⁶Since we put the brane at a fixed point X_0 , X_0 can "transform" to $\theta^{i+j} X_0 = X_0$, so i can be different from $-j$.

⁷Note that for the projective plane, if $i \neq -j$ then $X = X_0$ at $|z| = 1$.

where all the dependence on the sign \pm of the twist at z_2, \bar{z}_2 and some of the dependence on ϵ_2 is hidden the definition of x , as given in (2.20). The sum in (2.30) is over the whole coset lattice; the rescaling by 2 for $\epsilon_1 = -1$ is encoded in the prefactor $(5 - 3\epsilon_1)/2$ and the prefactor $(5 - 3\epsilon_1\epsilon_2)/2 = k^2$ (where k^2 is defined in (2.13)). We have also rescaled $v \rightarrow 2\pi Rv$; that is, the cycles of the torus are taken to have physical length $2\pi R$, but we define the generators of Λ to have length 1.

2.4.4 Relation to Dixon et. al.

To see how this relates to the calculation of the four point function on the sphere [4], recall that in that work the monodromy condition for the classical piece X_{cl} is

$$\Delta_{\mathcal{L}_i} X_{cl} = \oint_{\mathcal{L}_i} dz \partial X_{cl} + \oint_{\mathcal{L}_i} d\bar{z} \bar{\partial} X_{qu} = v_i,$$

where $i \in \{1, 2\}$, the loop \mathcal{L}_1 (\mathcal{L}_2) encloses the points w_2 and w_1 (w_2 and w_3) as defined in [4], and v_i runs over a coset lattice which depends on which twist fields are enclosed by \mathcal{L}_i .

If we set

$$(w_1, w_2, w_3, w_4) = (z_1, z_2, \epsilon_2/\bar{z}_2, \epsilon_2/\bar{z}_1) \tag{2.31}$$

for the case of H_{-+} , and

$$(w_1, w_2, w_3, w_4) = (z_1, \epsilon_2/z_2, z_2, \epsilon_2/\bar{z}_1) \tag{2.32}$$

for the case of H_{--} , then we have just learned that in our case

$$\Delta_{\mathcal{L}_i} X = 0,$$

where in the notations of [4] we use in this section $i = (3 + \epsilon_1)/2$.

Once we have the classical solution, we must remember to integrate it over $|z| \leq 1$ and not over the entire complex plane. We conclude that, if we define x as in (2.20) (see equation (4.37)), S_{cl} is $\frac{1}{2} \frac{5-3\epsilon_1}{2} \frac{5-3\epsilon_1\epsilon_2}{2}$ times the one in [4], where for $\epsilon_1 = 1$ $v_2 = 0$, $\epsilon_1 = -1$ for $v_1 = 0$ and when the argument of F is bigger than 1, we replace it with \tilde{F} (see appendix A).

Alternately, one can check that the most general classical solution given in [4] respects (2.3) only when $v_2 = 0$ for $\epsilon_1 = 1$ and $v_1 = 0$ for $\epsilon_1 = -1$.

3. Fermionic correlation functions

Twist fields for fermions can be constructed via bosonization, and *can* be factorized into holomorphic and antiholomorphic parts. Thus the correlators are simply computed via the doubling trick, without need of the conformal Ward identities.

Let us first review the twist fields and the bosonization map. The twist fields s_{\pm} are defined by the OPEs:

$$\begin{aligned}
 \psi(z)s_+(w) &\sim (z-w)^{k/N}t'_+(w) \\
 \psi^*(z)s_+(w) &\sim (z-w)^{-k/N}t_+(w) \\
 \bar{\psi}(\bar{z})\bar{s}_+(\bar{w}) &\sim (\bar{z}-\bar{w})^{-k/N}\bar{t}_+(\bar{w}) \\
 \bar{\psi}^*(\bar{z})\bar{s}_+(\bar{w}) &\sim (\bar{z}-\bar{w})^{k/N}\bar{t}'_+(\bar{w}) .
 \end{aligned} \tag{3.1}$$

The explicit construction of s_{\pm} shows that it can be so factorized into holomorphic and antiholomorphic pieces. Begin with the bosonization formula for ψ :⁸

$$\begin{aligned}
 \psi(z) &= e^{iH(z)} ; \psi^*(z) = e^{-iH(z)} \\
 \bar{\psi}(\bar{z}) &= e^{i\bar{H}(\bar{z})} ; \bar{\psi}^*(\bar{z}) = e^{-i\bar{H}(\bar{z})} .
 \end{aligned} \tag{3.2}$$

With this definition, the twist fields are simply defined:

$$s_{\pm}(z) = e^{\pm ikH(z)/N} ; \bar{s}_{\pm}(\bar{z}) = e^{\mp ik\bar{H}(\bar{z})/N} . \tag{3.3}$$

Given this definition, the excited fermionic twist fields t_{\pm}, t'_{\pm} , and so on, are simple to construct as well. Note that if ψ has left-moving R charge 1, s_{\pm} has fractional left-moving R charge $\pm k/N$.

To describe the doubling trick for the orbifolds we study, we need the action of (2.1) on the fermionic superpartners of X , and on the twist fields. In the case of $\epsilon_1 = -1$, worldsheet supersymmetry requires that we combine the action of (2.1) with a Z_2 subgroup of the $U(1)_R$ symmetry of the model. The resulting action of (2.1) on ψ is:

$$\psi(z) = \epsilon_1 \bar{\psi}(\bar{u}) \left(\frac{\partial \bar{u}}{\partial z} \right)^{\frac{1}{2}} \Big|_{\bar{u}=\epsilon_2/z} = \epsilon_1 \bar{\psi}(\epsilon_2/z) \left(\frac{-\epsilon_2}{z^2} \right)^{\frac{1}{2}} \tag{3.4}$$

for $|z| > 1$. The action of (2.1) on the twist fields must, by consistency, be:

$$\begin{aligned}
 s_{\pm}(z) &= e^{\pm i(1-\epsilon_1)\pi k/2N} \bar{s}_{\mp}(\bar{u}) \left(\frac{\partial \bar{u}}{\partial z} \right)^{\frac{k^2}{2N^2}} \Big|_{\bar{u}=\frac{\epsilon_2}{z}} \\
 \bar{s}_{\pm}(\bar{z}) &= e^{\mp i(1-\epsilon_1)\pi k/2N} s_{\mp}(u) \left(\frac{\partial \bar{u}}{\partial z} \right)^{\frac{k^2}{2N^2}} \Big|_{u=\frac{\epsilon_2}{\bar{z}}} ,
 \end{aligned} \tag{3.5}$$

where $\Delta = \frac{k^2}{2N^2}$ is the holomorphic (antiholomorphic) dimension of s_{\pm} (\bar{s}_{\pm}).

The relevant correlators for section 4 are, at this point, simple to calculate:

$$\langle s_-(z_1)\bar{s}_-(\bar{z}_1)s_{\pm}(z_2)\bar{s}_{\pm}(\bar{z}_2) \rangle \Big|_{z_1=0, z_2=y \in \mathbf{R}} = e^{\mp(1-\epsilon_1)i\pi k/N} |y|^{\mp 2k^2/N^2} |1 - \epsilon_2 y^2|^{-k^2/N^2} . \tag{3.6}$$

⁸Here we suppress the cocycles, as they will not be relevant for our computation.

4. Twisted sector kinetic terms to $\mathcal{O}(g_s)$ in type I on T^6/\mathbb{Z}_3

In this section we will describe an application of our results to the calculation of $\mathcal{O}(g_s)$ (disk and RP^2) corrections to the kinetic terms of certain twisted sector blow-up modes, for the T^6/\mathbb{Z}_3 compactification of type I string theory described in [5]. In section 4.1 we will describe the orbifold and the effective 4-dimensional picture for our calculation. In section 4.2 we will construct the blow-up mode vertex operator. In section 4.3 we will compute the various pieces of the amplitude coming from the disk and RP^2 with twist-twist and twist-anti twist insertions. In section 4.4 we will join the various pieces together by closely studying the spacetime physics of the boundaries of the moduli space of CFT correlators. In section 4.5 we will write the full amplitude for the type I orbifold, and further discuss the spacetime physics.

4.1 The T^6/\mathbb{Z}_3 orbifold and the 4d effective action

We consider the type I orbifold on T^6/\mathbb{Z}_3 , as described by [5, 7]. We parameterize the six-torus T^6 by three complex coordinates $X_{1,2,3}$, subject to the periodicity conditions

$$X_j \sim X_j + 2\pi R \sim X_j + 2\pi R\alpha, \quad (4.1)$$

where $\alpha = e^{2\pi i/3}$. This torus has a \mathbb{Z}_3 symmetry, under which the complex coordinates X_j transformed as

$$(X_1, X_2, X_3) \rightarrow (\alpha X_1, \alpha X_2, \alpha X_3). \quad (4.2)$$

The transformation (4.2) has 27 fixed points, located at $X_i = m_i (2\pi R e^{i\pi/6})/\sqrt{3}$, with $m_i = 0, 1, 2$. The resulting orbifold is a singular limit of a Calabi-Yau manifold with Euler character $\xi = 72$. The type II and heterotic versions of this orbifold were constructed in [8] and its geometry was analyzed in detail in [9]. Each of the 27 fixed points has a blow-up mode which leads to a modulus of the 4-dimensional effective field theory at closed string tree level, and which resolves the orbifold singularity; that remains true for the type I theory. The vertex operators for these blow-up modes are in the twisted sector of the corresponding fixed points. We may consider $C_k = \zeta_k + ic_k$ to be a closed string chiral multiplet. The real part ζ_k , is the NS-NS twist field which blows up the orbifold; in the type I orbifold, the imaginary part c_k is a Ramond-Ramond axion.

This theory has no D5-branes; the orbifold projects the type I gauge group down to $U(12) \times SO(8)$. The $U(1)$ factor is anomalous; the anomaly is cancelled by a 4d version of the Green-Schwarz mechanism. This requires that the Ramond-Ramond axion c in the symmetric combination $C = \frac{1}{\sqrt{27}} \sum_k C_k = \zeta + ic$ transforms by a shift under the open string $U(1)$. The result is that the Kähler potential of the low-energy theory can be written as:

$$\mathcal{K} = \mathcal{K}(C + \bar{C} - g_s V) \quad (4.3)$$

where V is the vector superfield for the anomalous $U(1)$. Note the relative factor of $g_s = \langle e^\phi \rangle$ in the gauging of the shift symmetry of c (here ϕ is the dilaton). This ensures that the tree-level couplings of the form $\frac{c}{g_s} \text{Tr} F \wedge F$ cancel the anomalous transformation of the path integral at one loop.

The original motivation for this work was to study the one-loop correction to the Fayet-Iliopoulos D-term for this U(1). We will discuss this further in [10]; here we note that in many string models this controls the tension of interesting cosmic D-strings [11, 12], and controls the decay constant of some interesting candidates for QCD axions [13].

The one-loop correction arises from an $\mathcal{O}(g_s)$ correction to the Kähler potential for X . To see this, note that to one-loop order in g_s and to quadratic order in C the $d^4\theta$ terms in the $N = 1$ theory should have the form:

$$K = \frac{1}{g_s^2}(1 + bg_s + dg_s^2)(C + \bar{C} - g_s V)^2 + \frac{1}{g_s}(a + cg_s)V + \dots \quad (4.4)$$

Here, a, b, c, d are numerical coefficients, and we have suppressed the fluctuations of the dilaton. Explicit calculations [14, 7] have shown that $a = c = 0$. In other words, to one loop order, the Fayet-Iliopoulos D-term vanishes at the orientifold point. The tree-level kinetic term for C induces a field-dependent FI D-term at open string tree level [14], which is nonzero for the blown-up orbifold.

It had been conjectured in [7, 15] (and stated in various works since) that the one-loop correction to the field-dependent D-terms would vanish away from the orientifold point. In particular this would imply that $b = 0$. However, from an effective field theory standpoint, we see no reason for b to vanish. In the remainder of this section we will calculate this term and find that it is in fact non-vanishing.

We will extract b from the $\mathcal{O}(k^2)$ part of the two-point functions of twist fields on the disk and RP^2 . For truly massless fields the on-shell condition is $k^2 = 0$. However, the FI D-term above gives a positive $\mathcal{O}(g_s)$ mass to a linear combination of C and the charged scalars. More precisely, the kinetic term for the anomalous U(1) gauge field is:

$$L = V_6 \int d^2\theta \left(\frac{\alpha}{g_s} + \Delta \right) \mathcal{W}_\alpha \mathcal{W}^\alpha + \dots \quad (4.5)$$

where \mathcal{W}_α is the chiral Fermi superfield for the gauge multiplet, $g_{YM,tree} = \sqrt{g_s/\alpha}$ is the tree-level gauge coupling (which will also generically depend on C), Δ is the one-loop threshold correction to the gauge coupling and V_6 is the volume of T^6/\mathbb{Z}_3 in string units. If we integrate out the auxiliary D field in the gauge multiplet, the quadratic action for ζ becomes:

$$I \sim \frac{1}{g_s^2} \int dx^4 \left[\partial_\mu \zeta \partial^\mu \zeta (1 + bg_s + dg_s^2) - \frac{g_s}{V_6} (1 + (2b - \Delta)g_s + \dots) \zeta^2 \right], \quad (4.6)$$

String theory amplitudes compute amputated on-shell matrix elements. The two-point function thus vanishes on shell, as it is proportional to the inverse propagator. In the present case, the closed string field C gets a mass from disk and RP^1 diagrams. Thus the contribution of any given loop order will not vanish; rather, the full two-point function vanishes through cancellations between different loop orders. In particular the on-shell condition will be $k^2 \sim g_s + \dots$. We will therefore compute b by computing the $\mathcal{O}(k^2)$ term for the two-point function on the disk and RP^2 .

4.2 The closed string vertex operator

The first step is to construct the vertex operators corresponding to the blow-up mode of that fixed point. Let $\mathcal{X}^{\mu=0,1,2,3}$ parameterize the non-compact dimensions (4d Minkowski space) transverse to the orbifold; c, \bar{c} be the anticommuting conformal ghosts and $\phi, \bar{\phi}$ arise from the bosonization of the superconformal ghosts (see for example [16, 17]). The vertex operator for the massless blow-up mode in the $(-1, -1)$ picture is

$$V_{\pm 1/3}^{(-1,-1)}(z, \bar{z}; k) = \frac{1}{\sqrt{27}} e^{-\phi(z) - \bar{\phi}(\bar{z})} e^{ik_\mu \mathcal{X}^\mu} \prod_{i=1}^3 \left(\sum_{p=0}^2 \sigma_{p,\pm}^{(i)}(z, \bar{z}) s_{\pm}^{(i)}(z) \bar{s}_{\pm}^{(i)}(\bar{z}) \right), \quad (4.7)$$

where $k^2 = 0$. The sum over p is a sum over the fixed points of T^2/\mathbb{Z}_3 : since fixed points of T^6/\mathbb{Z}_3 are in particular fixed points of $(T^2/\mathbb{Z}_3)^3$, the twist operators in (4.7) are the product of the corresponding ones in each T^2/\mathbb{Z}_3 factor.

The orientifold action maps twisted sectors to their anti-twist sectors, breaking the \mathbb{Z}_3 quantum symmetry of the type IIB orbifold [18] that would describe the closed string sector absent the D9 branes and O9 planes of type I string theory. The orientifold action interchanges $V_{1/3}$ and $V_{-1/3} \equiv \Omega V_{1/3} \Omega$. Therefore only the combination

$$V^{(-1,-1)} = V_{1/3}^{(-1,-1)} + \Omega V_{1/3}^{(-1,-1)} \Omega \equiv V_{1/3}^{(-1,-1)} + V_{-1/3}^{(-1,-1)} \quad (4.8)$$

survives the orientifold projection.

4.3 Computing the two-point amplitudes

The total ϕ ($\bar{\phi}$) charge on a worldsheet h holes plus crosscaps and g handles is $2g + h - 2$. For the disk and RP^2 , $g = 0$ and $h = 1$. So we will take one vertex operator to be in the $(-1, -1)$ picture and the other in the $(0, 0)$ picture. The $SL(2, \mathbb{R})$ symmetry of these amplitudes allows us to fix the position of the two vertex operators to $(z_1, z_2) = (0, y)$, where $y \in [0, 1]$.

The D-branes and orientifold planes in type I string theory fill the 10d spacetime, and so corresponds to the case $\epsilon_1 = 1$ in section 2.

The amplitudes on the disk should include a trace over Chan-Paton factors. For correlation functions on the disk, an element g of the orbifold group Γ acts on the Chan-Paton matrices t as in [19, 14], by conjugation: $t \rightarrow \gamma_h t \gamma_g^{-1}$, where γ_g is the image of a group homomorphism of Γ into the open string gauge group. For specificity, the matrices $\gamma_{\frac{1}{3}}$ for the \mathbb{Z}_3 twists are given in [7].

In order to compute the $\mathcal{O}(g_s)$ correction to the kinetic term for blowup modes, we must compute the disk correlators

$$\begin{aligned} \mathcal{A}_{\pm, \pm}^{\epsilon_2=1} &= \text{Tr} \left(\gamma_{\pm \frac{1}{3}} \gamma_{\pm \frac{1}{3}} \right) \times \\ &\times \int_0^1 dy \langle \xi \left(c \bar{c} e^{\phi} T_F e^{\bar{\phi}} \bar{T}_F V_{\pm \frac{1}{3}}^{(-1,-1)}(0, 0; k) \right) \left((c - \bar{c}) V_{\pm \frac{1}{3}}^{(-1,-1)}(y, y; -\tilde{k}) \right) \rangle. \end{aligned} \quad (4.9)$$

and the RP^2 correlators which are identical but for the absence of the Chan-Paton trace and for an additional normalization factor which we will determine in section 4.4. Here ξ is the zero mode in the superconformal ghost sector as described in [16, 17].

The fermionic stress tensor is

$$T_F(z) = -\frac{1}{4} \sum_{i=1}^3 (\partial X_i \psi_i^* + \partial \bar{X}_i \psi_i) - \frac{1}{2} \partial \mathcal{X} \cdot \Psi, \quad (4.10)$$

with a similar definition for $\bar{T}_F(\bar{z})$. $e^\phi T_F V(z)$ denotes the simple pole in the OPE of $e^\phi T_F(w) V(z)$, and is the result of acting with the picture changing operator on V . Note that for the disk and RP^2 , only the sum of left- and right-moving ghost number is conserved. Therefore, we can equally well do the computation with both vertex operators in the $(-1, 0)$ picture, or with one vertex operator in the $(-1, 0)$ picture and the other in the $(0, -1)$ picture.

We first compute

$$N_{-+}^{\epsilon_2} \int_0^1 dy \langle \xi \left(c\bar{c} e^\phi T_F V_{1/3}^{(-1,-1)}(0, 0; k) \right) \left((c-\bar{c}) e^{\bar{\phi}} \bar{T}_F V_{-1/3}^{(-1,-1)}(y, y; -\tilde{k}) \right) \rangle \equiv \int_0^1 dy \mathcal{H}_{-+}^{\epsilon_2}(y). \quad (4.11)$$

where $N_{-+}^{\epsilon_2}$ is a normalization factor. Following [4], we can simplify the amplitude by examining the action of the picture-changing operators more closely. In the OPE of the compact part of T_F with $\sigma_+^{(i)} s_+^{(i)}$, only the $\partial X_i \psi_i^*$ term is singular. Similarly, in the OPE of the compact part of \bar{T}_F with $\sigma_-^{(i)} \bar{s}_-^{(i)}$, only the $\partial \bar{X}_i \bar{\psi}_i^*$ term is singular. The B-type boundary conditions we study here conserve the sum of the holomorphic and antiholomorphic fermion number [20]. Therefore, this part of the amplitude vanishes by fermion number conservation, and only the non-compact part of T_F (\bar{T}_F) involving $-\frac{1}{2} \partial \mathcal{X} \cdot \Psi$ ($-\frac{1}{2} \bar{\partial} \mathcal{X} \cdot \bar{\Psi}$) contributes to this amplitude. The nonvanishing part of $H_{-+}(y)$ is:

$$\begin{aligned} \mathcal{H}_{-+}^{\epsilon_2}(y) &= N_{-+}^{\epsilon_2} \sum_{p,q=1}^{27} \langle e^{-\bar{\phi}}(0) e^{-\phi}(y) \rangle \langle c(0) \bar{c}(0) [c(y) - \bar{c}(y)] \rangle \\ &\quad \times \langle e^{ik \cdot \mathcal{X}}(0) e^{-i\tilde{k} \cdot \mathcal{X}}(y) \rangle \langle k \cdot \Psi(0) (-\tilde{k}) \cdot \bar{\Psi}(y) \rangle \\ &\quad \times \langle \sigma_{+,p}(0, 0) \sigma_{-,q}(y, y) \rangle \langle s_+(0) \bar{s}_+(0) s_-(y) \bar{s}_-(y) \rangle, \end{aligned} \quad (4.12)$$

where $\sigma_{\pm,p}$ and s_{\pm} are shorthands for the products of $\sigma_{\pm}^{(i)}$ and $s_{\pm}^{(i)}$ over the three T^2 factors of T^6 , and p, q label the orbifold fixed points. Note that the term $\langle e^{ik \cdot \mathcal{X}}(0) e^{-i\tilde{k} \cdot \mathcal{X}}(y) \rangle \langle k \cdot \Psi(0) (-\tilde{k}) \cdot \bar{\Psi}(y) \rangle$ is proportional to k^2 .

The second term in (4.10) that we have to compute is

$$N_{--}^{\epsilon_2} \int_0^1 dy \langle \left(c\bar{c} e^\phi T_F V_{-1/3}^{(-1,-1)}(0, 0; k) \right) \left((c-\bar{c}) e^\phi T_F V_{-1/3}^{(-1,-1)}(y, y; -\tilde{k}) \right) \rangle \equiv \int_0^1 dy \mathcal{H}_{--}^{\epsilon_2}(y), \quad (4.13)$$

and similarly for $H_{++}(y)$. As for H_{-+} , when choosing the appropriate picture, only the 4d spacetime parts of T_F contribute to the amplitude. The result is

$$\begin{aligned} \mathcal{H}_{--}^{\epsilon_2}(y) &= N_{--}^{\epsilon_2} \langle e^{-\phi}(0) e^{-\phi}(y) \rangle \langle c(0) \bar{c}(0) [c(y) - \bar{c}(y)] \rangle \\ &\quad \times \langle e^{ik \cdot \mathcal{X}}(0) e^{-i\tilde{k} \cdot \mathcal{X}}(y) \rangle \langle k \cdot \bar{\Psi}(0) (-\tilde{k}) \cdot \bar{\Psi}(y) \rangle \\ &\quad \times \langle \sigma_-(0, 0) \sigma_-(y, y) \rangle \langle s_-(0) \bar{s}_-(0) s_-(y) \bar{s}_-(y) \rangle. \end{aligned} \quad (4.14)$$

The results for \mathcal{H}_{+-} and \mathcal{H}_{++} will be identical to the above, and we will not discuss them explicitly.

4.3.1 The ghost and spacetime parts of the amplitudes

The contributions of the ghosts and spacetime parts are:

$$\begin{aligned}
 \langle e^{-\phi(0)} e^{-\phi(y)} \rangle &= 1/y \\
 \langle e^{-\bar{\phi}(0)} e^{-\phi(y)} \rangle &= \omega_\phi \\
 \langle k \cdot \Psi(0)(-\tilde{k}) \cdot \bar{\Psi}(y) \rangle &= \omega_\psi k \cdot \tilde{k} \\
 \langle k \cdot \Psi(0)(-\tilde{k}) \cdot \Psi(y) \rangle &= k \cdot \tilde{k}/y \\
 \langle c(0)\bar{c}(0)[c(y) - \bar{c}(y)] \rangle &= \omega_c y \\
 \langle e^{ik \cdot \mathcal{X}(0,0)} e^{-i\tilde{k} \cdot \mathcal{X}(y,y)} \rangle &= \omega_\mathcal{X} \delta^4(k - \tilde{k}) \left[\frac{1 - \epsilon_2 y^2}{y^2} \right]^{k^2}.
 \end{aligned} \tag{4.15}$$

Here $\omega_{\phi,\psi,c,\mathcal{X}}$ are phase factors which we will set later using spacetime considerations, $\delta^4(k - \tilde{k})$ comes from integration over the zero modes in \mathbb{R}^4 and is a volume divergence.

4.3.2 Twist field correlators and the coset lattice

The correlation function of twist fields can be broken up following (2.10) into a classical and a quantum piece. The quantum part of the correlator is independent of the particular fixed point on which the classical solution resides.

To find the right lattice, note that the two-point function of Φ breaks up into a sum over fixed points.

$$\langle \Phi \Phi \rangle = \frac{1}{27} \sum_{p,q} \langle \Phi_p \Phi_q \rangle = \sum_q \langle \Phi_p \Phi_q \rangle \tag{4.16}$$

where the last line is p -independent. As shown in section 3 of [4] (c.f. figure 3 of that work), the sum over q and the coset lattice (2.28) is just the sum over the full Narain lattice Λ . We can define this lattice as:

$$\Lambda = \{ \eta(n_1, n_2, n_3) + \bar{\eta}(m_1, m_2, m_3) \mid n_i, m_i \in \mathcal{Z}, \eta = \frac{1}{\sqrt{3}}(1 + e^{i\pi/3}) \}. \tag{4.17}$$

Once we have this, the results of section 2 and section 3 for $\epsilon_1 = 1$ give us the twist field contributions to the two-point function.

4.3.3 The full amplitude

The full amplitude is:

$$\begin{aligned}
 \mathcal{H}_{-\pm}^{\epsilon_2}(y) &= N_{-\pm}^{\epsilon_2} \omega_{\text{tot}} \frac{k^2}{y} \frac{1}{1 - \epsilon_2 y^{\pm 2}} \left[\frac{1 - \epsilon_2 y^{\pm 2}}{y^{\pm 2}} \right]^{k^2} \delta^4(k - \tilde{k}) \\
 &\times \left[\tilde{F}^{-1}(\epsilon_2 y^{\pm 2}) \sum_{m,n=-\infty}^{\infty} \exp \left(-\frac{\pi R^2 (5 - 3\epsilon_2)}{12\sqrt{3}\alpha'} \frac{\tilde{F}(1 - \epsilon_2 y^{\pm 2})}{\tilde{F}(\epsilon_2 y^{\pm 2})} |n\eta + m\bar{\eta}|^2 \right) \right]^3.
 \end{aligned} \tag{4.18}$$

Here the phase ω_{tot} is the product of all of the phases $\omega_{\phi,\psi,c,\mathcal{X}}$.

We wish to add together all of the amplitudes $\int_0^1 dy \mathcal{H}_{\pm,\pm}^{\epsilon_2}$. We will do so in the next sub-section using tadpole cancellation arguments.

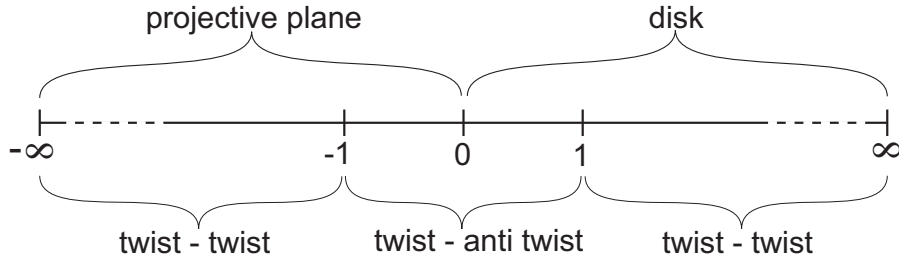


Figure 6: The different piece of the x -integral. $x \in [-\infty, -1]$ comes from the projective plane with two identical twist insertions, $x \in [-1, 0]$ comes from the projective plane with twist and anti-twist insertions, $x \in [0, 1]$ comes from the disk with twist and anti-twist insertions, $x \in [1, \infty]$ comes from the disk with two identical twist insertions.

4.4 Connecting the pieces

Define

$$\begin{aligned} \tilde{\mathcal{H}}_{-\pm}^{\epsilon_2}(x) &\equiv \frac{1}{\delta^4(k - \tilde{k})} \mathcal{H}_{-\pm}^{\epsilon_2}(y) \frac{dy}{dx} \\ &= \mathcal{N}_{-\pm}^{\epsilon_2} \frac{k^2}{x(1-x)} \left[\frac{1-x}{x} \right]^{k^2} \left[\tilde{F}^{-1}(x) \sum_{m,n=-\infty}^{\infty} \exp\left(-\frac{\pi R^2(5-3\epsilon_2)}{4\sqrt{3}\alpha'} \frac{\tilde{F}(1-x)}{\tilde{F}(x)} |n\eta + m\bar{\eta}|^2 \right) \right]^3. \end{aligned} \quad (4.19)$$

Here \mathcal{N} is the product of all numerical factors. The variable x is defined in (2.20) — its range depends on the relative sign of the twist fields, and on the boundary conditions, as shown in figure 6. The sole dependence of the amplitude on the relative signs of the twist fields is encoded in the range of x .

As a result, we can consider the full amplitude — the sum of the modular integrals of the amplitudes over ϵ_2 and over the signs of the twist fields — to be a modular integral of a single integrand over the entire real x -axis. The integrand is one of the functions $\mathcal{H}_{\pm,\pm}^{\epsilon_2}$ depending on the value of x . Note that for fixed ϵ_2 and sign of the twist fields, $\mathcal{H}_{\pm,\pm}^{\epsilon_2}$ are not necessarily finite at the boundaries of the range in x over which they are defined. However, the type I model in T^6/\mathbb{Z}_3 is known to be a consistent string background. Therefore, any divergences must either cancel in the full amplitude, or correspond to IR divergences arising from the exchange of physical massless modes. We will use these facts to fix the relative values of $\mathcal{N}_{\pm,\pm}^{\epsilon_2}$.

We begin by noting that the amplitudes for $\mathcal{H}_{-\pm}$ and $\mathcal{H}_{+\mp}$ will give identical contributions since

$$\langle V_{-1/3} V_{\pm 1/3} \rangle = \langle V_{-1/3} \Omega V_{\mp 1/3} \Omega \rangle = \langle \Omega V_{-1/3} \Omega V_{\mp 1/3} \rangle = \langle V_{1/3} V_{\mp 1/3} \rangle. \quad (4.20)$$

Therefore, we need only determine four amplitudes, $\mathcal{H}_{-\pm}^{\epsilon_2=\pm 1}$.

Now let us discuss in turn the behavior near $x = -1, 0, \pm\infty$, and 1, at which we must join together $\mathcal{H}_{-\pm}^{\epsilon_2}$.

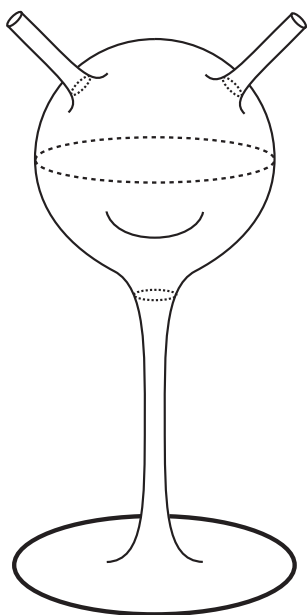


Figure 7: At one boundary of the moduli space, when the two closed string vertex operators approach each other, the amplitude factorizes onto a three-point function on the sphere times a tadpole diagram, here drawn for the disk.

4.4.1 The amplitudes near $x = -1$ — continuity constraint

The first relation between the coefficients $\mathcal{N}_{-\pm}^-$ arises from the fact that $x = -1$ is not actually a boundary of the moduli space of the corresponding correlation function. More precisely, the RP^2 amplitudes $\tilde{\mathcal{H}}_{--}^-(-\infty \leq x \leq -1)$ and $\tilde{\mathcal{H}}_{-+}^-(-1 \geq x \leq 0)$ connect continuously at $x = -1$, as $V_{1/3}$ at $|z| = 1$ is equal to $\Omega V_{1/3} \Omega = V_{-1/3}$ at $z = -1/\bar{z}$. Therefore, $\tilde{\mathcal{H}}_{--}^-(-1)dx$ and $\tilde{\mathcal{H}}_{-+}^-(-1)dx$ are the *same* amplitude, which implies that:

$$\frac{\mathcal{N}_{--}^-}{\mathcal{N}_{-+}^-} = 1, \tag{4.21}$$

4.4.2 The amplitude near $x = 0, \pm\infty$ — tadpole cancellation constraints

The remaining points $x = 0, \pm\infty, 1$ are true boundaries of moduli space of 2d correlators on the disk. At $x \rightarrow 0, \infty$ the two closed string vertex operators approach each other and the amplitude factorizes onto a three-point function on the sphere times a tadpole diagram, joined by the propagator for the massless closed string mode (see figure 7). In these limits the x integral diverge.⁹ Since all tadpoles cancel for the type I model we are studying [5], the disk and RP^2 contributions should be added such that the divergences due to tadpoles cancel.

For $\mathcal{H}_{--}^{\epsilon_2}$, the RP^2 tadpole at $x \rightarrow -\infty$ and the disk tadpole at $x \rightarrow \infty$ should cancel.¹⁰ This requires:

$$\frac{\mathcal{N}_{--}^-}{\mathcal{N}_{-+}^+} = -8, \tag{4.22}$$

where we have used the asymptotic behaviors:

$$F(-x) \sim \frac{\Gamma(1/3)}{\Gamma^2(2/3)} x^{-1/3}, \quad \tilde{F}(x) \sim \frac{1}{2} \frac{\Gamma(1/3)}{\Gamma^2(2/3)} x^{-1/3}, \quad x \rightarrow \infty. \tag{4.23}$$

For $\mathcal{H}_{-+}^{\epsilon_2}$, the divergence in the disk tadpole as $x \rightarrow 0^+$ should cancel the divergence in the RP^2 tadpole as $x \rightarrow 0^-$. This requires that

$$\frac{\mathcal{N}_{-+}^-}{\mathcal{N}_{-+}^+} = 1, \tag{4.24}$$

⁹Remember that $k^2 \sim g_s > 0$.

¹⁰The nonvanishing $(--)$ amplitudes at $\mathcal{O}(g_s)$ follow from the breaking of the quantum symmetry of the oriented closed string theory. The orientifold already breaks the quantum symmetry, as the orientifold action maps twisted sectors to their conjugates [18]. Furthermore, the D-branes can also break this quantum symmetry, as can be seen from the existence of twisted sector tadpoles that must be cancelled with the correct choice of gauge group [19, 5].



Figure 8: At one boundary of moduli space, when one of the closed string operators approach the boundary of the disk, the disk amplitude factorize into two disks with closed string twist insertions, connected by a massless open string propagator.

where we have used the asymptotic behaviors:

$$F(x) \sim 1, \quad F(1 - |x|) \sim \tilde{F}(1 + |x|) \sim -\frac{\sin(\pi k/N)}{\pi} \ln(x), \quad x \sim 0. \quad (4.25)$$

4.4.3 The amplitude near $x = 1$ and open string dynamics

The final boundary of moduli space is at $x = 1$. The amplitudes $\mathcal{H}_{-\pm}^+$ factorize in the limit $x \rightarrow 1^\mp$ onto two closed string one-point function on the disk, joined by an open string propagator (see figure 8). The relative weights of these amplitudes are completely fixed by (4.21), (4.22) and (4.24) to be:

$$\frac{\mathcal{N}_{-+}^+}{\mathcal{N}_{--}^+} = -8, \quad (4.26)$$

We would like to check that this is consistent.

First, we note that (4.26) is completely consistent with the Chan-Paton factors. By doing Poisson resummations of the sums in (4.19), using the formula

$$\beta \sum_{n,m} e^{-\frac{\pi\beta}{\tau_2}|m-\tau n|^2} = \sum_{k,w} e^{-\frac{\pi}{\beta\tau_2}|k-\tau w|^2}, \quad (4.27)$$

we can see that \mathcal{H}_{--}^+ and \mathcal{H}_{-+}^+ , have the same behavior near $x = 1$, with their normalization differing only by the Chan-Paton factors. These are $\text{Tr}(\gamma_- \gamma_-)$ and $\text{Tr}(\gamma_- \gamma_+)$, respectively. Using the explicit expressions in [7] for γ_\pm :

$$\gamma_{\pm\frac{1}{3}} = \begin{pmatrix} \mathbf{1}_8 & 0 & 0 \\ 0 & -\frac{1}{2}\mathbf{1}_{12} & \pm\frac{\sqrt{3}}{2}\mathbf{1}_{12} \\ 0 & \mp\frac{\sqrt{3}}{2}\mathbf{1}_{12} & -\frac{1}{2}\mathbf{1}_{12} \end{pmatrix}, \quad (4.28)$$

where the $\mathbf{1}_d$ is the $d \times d$ identity matrix, we find that:

$$\frac{\text{Tr}(\gamma_- \gamma_+)}{\text{Tr}(\gamma_- \gamma_-)} = -8 \quad (4.29)$$

in agreement with (4.26).

The explicit behavior of the amplitude near $x = 1^\pm$ is:

$$\tilde{\mathcal{H}}_{-, \mp} \sim \frac{k^2/R^6}{(1-x)^{1+k^2}}, \quad (4.30)$$

where the R^6 comes from the Poisson resummation. Thus, although the amplitude appears to be proportional to k^2 , the formula

$$\lim_{k^2 \rightarrow 0} \frac{k^2}{(1-x)^{1+k^2}} = \delta(1-x), \tag{4.31}$$

shows that there is a leading term which is constant as $k^2 \rightarrow 0$: it is an $\mathcal{O}(g_s)$ mass term. The $1/R^6$ in (4.30) matches the $1/V_6$ factor in (4.6).

Such a term could arise from integrating out the massless (at leading order in g_s) U(1) gauge field if it has a derivative coupling to the physical blow-up mode (the mode that survives the orientifold projection). However, this coupling vanishes. The disk amplitude for this open-closed coupling is

$$A_{\pm} = \langle V_{\pm\frac{1}{3}}(0,0;k) \partial \mathcal{X}^{\mu}(1) e^{-ik \cdot \mathcal{X}}(1) \zeta_{\mu} \rangle \left(\text{Tr} \gamma_{\pm\frac{1}{3}} t_{\text{U}(1)} \right) \propto k \cdot \zeta \text{Tr} \left(\gamma_{\pm\frac{1}{3}} t_{\text{U}(1)} \right), \tag{4.32}$$

where ζ_{μ} is the polarization vector and $t_{\text{U}(1)}$ is the U(1) Chan-Paton matrix. The only difference between A_+ and A_- is the Chan-Paton trace. Using $\gamma_{\pm\frac{1}{3}}$ (4.28) and t as given in [7]:

$$t_{\text{U}(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{12} \\ 0 & -\mathbf{1}_{12} & 0 \end{pmatrix}, \tag{4.33}$$

we find that

$$\text{Tr} \left(\gamma_{\frac{1}{3}} t_{\text{U}(1)} \right) = -\text{Tr} \left(\gamma_{-\frac{1}{3}} t_{\text{U}(1)} \right). \tag{4.34}$$

Thus (4.8) does not couple to any propagating massless fields.

In fact, the origin of this term is as a contact term in the CFT. Such contact terms have been discussed in the context of closed strings in [21, 22]. The interpretation here is the same — the contact term arises from integrating out the non-propagating auxiliary field in the U(1) $\mathcal{N} = 1$ supermultiplet. In this case, the open string channel can be thought of as corresponding to the auxiliary field, and the disk diagrams corresponds to tree-level dependence of the Fayet-Iliopoulos D-term on the twist fields, which is known to be nonvanishing, as is clear from (4.4) [5, 7].¹¹

As a check that the disk diagrams are nonvanishing, in light of our discussion of the vanishing of the scalar-gauge-field couplings above, note that the corresponding vertex operator is the U(1)_R current of the model on the boundary [21, 22, 15]. Since $V_{\pm\frac{1}{3}}$ have opposite worldsheet R-charge, the signs from the Chan-Paton traces (4.34) are cancelled by the signs from the R-charge sector. The Chan-Paton factors multiply a nonvanishing CFT amplitude.

4.5 The full amplitude on T^3/\mathbb{Z}_3

As discussed in section 4.1, we are expecting two distinct contributions to (4.6) from the disk and RP^2 diagrams. One is the $\mathcal{O}(g_s)$ mass for ζ (from integrating out the auxiliary D-field) and the other is the $-g_s k^2 b$ correction to the kinetic term for ζ . The on-shell

¹¹See also [23].

condition is $k^2 \sim g_s$ and our calculation should apply for any (small) value of g_s . Therefore the distinction between the two contributions to (4.6) are in the k^2 dependence of the disk and RP^2 diagrams, before we put k^2 on-shell.

To extract the $\mathcal{O}(g_s)$ mass term we put $k^2 = 0$. The $\mathcal{O}(g_s)$ mass term is reproduced in our amplitude from the contact term that arises as $k^2 \rightarrow 0$, in the region of integration near $x = 1$. It comes with the numerical coefficient

$$c = \frac{1}{2} (\mathcal{N}_{--}^+ + \mathcal{N}_{++}^+) + \frac{1}{2} (\mathcal{N}_{-+}^+ + \mathcal{N}_{+-}^+) = -7\mathcal{N}_{--}^+ . \quad (4.35)$$

To extract the correction to the kinetic term $(-g_s k^2 b)$, we hold k^2 fixed, subtract the $\mathcal{O}(g_s)$ mass term (4.35) and divide the result by k^2 . In order to extract the value of b we then take the limit $k^2 \rightarrow 0$ — only the contact term (4.31) made this order of limits problematic, and we have subtracted it out.

More explicitly

$$b_n \propto \lim_{k^2 \rightarrow 0} \frac{1}{k^2} \left[\int_1^\infty \tilde{\mathcal{H}}_{--}^+(x) dx + \int_0^1 \tilde{\mathcal{H}}_{-+}^+(x) dx + \int_{-\infty}^0 \tilde{\mathcal{H}}_{-+}^-(x) dx - c \right] \equiv \tilde{b} . \quad (4.36)$$

We do not know how to evaluate the integrals in (4.36) analytically. Note that (4.36) does not possess any obvious symmetry property and therefore, in appose to (4.35), we expect it to depend on R (the volume of T^6/\mathbb{Z}_3). Using Mathematica we evaluated it numerically in the non-compact limit ($R \rightarrow \infty$), where the classical contribution (2.30) become trivial.¹² We found the non-zero value for (4.36):¹³

$$\tilde{b}(R \rightarrow \infty) = 4.7 \pm .1 . \quad (4.37)$$

Since we expect $b(R)$ to have a smooth limit as $R \rightarrow \infty$, we take (4.37) as evidence that $b(R) \neq 0$ for generic R .

The FI D-term enters into various physical quantities such as the physical mass of ξ as shown in (4.6). Even though b is nonvanishing, one might wonder if $2b - \Delta$ in (4.6) vanishes at the orbifold point — indeed, in the type IIA model in [6], it was argued that the one-loop correction to the D-term is equal to the threshold correction at the orientifold point. In the present model this cannot be the case. In particular, the instanton corrections give b an explicit dependence on the untwisted Kähler moduli R . However, at the orbifold point the cylinder and Möbius strip amplitudes at constant field strength [25] from which one extracts the threshold correction Δ are independent of the untwisted Kähler moduli.¹⁴ Thus, we expect that for general R , neither the one-loop correction to the field-dependent FI D-term, nor the physical mass for ξ , will vanish.

¹²It is hard to maintain numerical precision in evaluating the hypergeometric function $F(x)$ near $x = 1$. To deal with this problem, we use its explicit asymptotic behavior (4.25) near $x = 1$, as explained in [24].

¹³The poor accuracy arises because we are canceling logarithmic divergences.

¹⁴The one-loop correction to the gauge coupling anomalous $U(1)$ is not computed in [25], as it requires an additional subtraction of divergences. Nonetheless, the cylinder and Möbius strip diagrams which must contribute to the answer *are* computed there, and they are independent of the untwisted Kähler moduli.

5. Conclusions

Since (4.37) is nonvanishing, (4.3) indicates that in type I compactification on T^6/\mathbb{Z}_3 there will be a correction to the Fayet-Iliopoulos D-term at order g_s and to first order in a perturbation of the theory away from the orientifold point. This is consistent with the explicit calculation in [7] which shows that the one-loop correction vanishes *at* the orientifold point. It contradicts the conjecture that this correction would vanish away from the orientifold point given in [7, 15], but there were no strong arguments for this conjecture, and no clear reason for it to hold from the standpoint of 4d effective field theory.

It is sometimes stated that the results of [26] imply that there should be no contributions at higher loops. However, these results depend on the field theory being renormalizable and on the gauge symmetry in the UV being linearly realized. These conditions are absent in our string model and in many others, and we see no reason for the nonrenormalization theorem to hold — in particular, the coefficient d in (4.4) leads to a field-dependent FI D-term at two loops in open string perturbation theory. From the standpoint of effective field theory, we see no reason for this term to vanish, and suspect that it does not. We will discuss the effective field theory point of view in more detail in [10].

Acknowledgments

We would like to thank Allan Adams, Howard Schnitzer and Eva Silverstein for helpful discussions. We are especially thankful to John McGreevy for many valuable discussions. Much of this work was carried out at the Kavli Institute for Theoretical Physics at UCSB, during the "String Phenomenology" workshop. We would also like to thank the MIT Center for Theoretical Physics for their generous hospitality during various parts of this project. A.L. would also like to thank the theory group at CEA Saclay for their generous hospitality as this work was completed. This research was supported in part by the National Science Foundation under Grant No. PHY99-07949. The research of A.L. and A.S. is further supported in part by NSF grant PHY-0331516, by DOE Grant No. DE-FG02-92ER40706, and by an Outstanding Junior Investigator award.

A. Definition of \tilde{F}

The hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z)$ has a branch point at $z = 1$; the branch cut can be taken along the positive real axis for $z > 1$. In this paper we find that it is natural to express the correlators in terms of a particular continuation of F to $z > 1$. We note that, while writing F for all values of z , [4] do not discuss such a continuation. The continuation below is completely consistent with their results.

Recall that an integral definition of the hypergeometric function (of the first kind) is

$${}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 \frac{t^{\beta-1}(1-t)^{\gamma-\beta-1}}{(1-tz)^\alpha} dt, \quad (\text{A.1})$$

where by a^b for a positive and b fractional, we mean the real positive branch. This integral is unambiguously defined for $|z| < 1$. At $z \sim 1$ it diverges logarithmically while for $z > 1$

one must decide how one continues past the branch point at $z = 1$. Our monodromy integrals are given in term of the sum of continuations above and below the branch cut. For $\alpha, \beta < 1$ (which is always the case here), that sum can be written as

$$\begin{aligned}
 & {}_2\tilde{F}_1(\alpha, \beta; \gamma; z > 1) \\
 & \equiv \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \left[\int_0^{1/z} \frac{t^{\beta-1}(1-t)^{\gamma-\beta-1}}{(1-tz)^\alpha} dt + \cos(\pi\alpha) \int_{1/z}^1 \frac{t^{\beta-1}(1-t)^{\gamma-\beta-1}}{(tz-1)^\alpha} dt \right] \\
 & = \frac{\Gamma(\gamma)\Gamma(1-\alpha)}{\Gamma(1-\alpha+\beta)\Gamma(\gamma-\beta)} z^{-\beta} {}_2F_1(1-\gamma+\beta, \beta; 1-\alpha+\beta; 1/z) \\
 & \quad + \cos(\pi\alpha) \frac{\Gamma(\gamma)\Gamma(1-\alpha)}{\Gamma(\gamma+1-\alpha-\beta)\Gamma(\beta)} \frac{z^{1-\gamma}}{(z-1)^{\alpha+\beta-\gamma}} {}_2F_1(1-\alpha, 1-\beta; 1-\alpha-\beta+\gamma; 1-z),
 \end{aligned} \tag{A.2}$$

and

$${}_2\tilde{F}_1(\alpha, \beta; \gamma; z \leq 1) = {}_2F_1(\alpha, \beta; \gamma; z \leq 1). \tag{A.3}$$

Equivalently, for any real value of z , one can think of ${}_2\tilde{F}_1$ as the limit of

$${}_2\tilde{F}_1(\alpha, \beta; \gamma; z) \equiv \frac{1}{2} [{}_2F_1(\alpha, \beta; \gamma; z) + {}_2F_1(\alpha, \beta; \gamma; \bar{z})], \tag{A.4}$$

for which z approaches the real line, and the integral in (A.1) is evaluated along the real line.

B. Some useful hypergeometric integrals

In calculating the monodromy integrals in section 2, some useful integrals for $y \in [0, 1]$ are

$$\int_0^y dx [x(1/y - \epsilon_2 x)]^{-\nu} (y-x)^{-(1-\nu)} = y^\nu \Gamma(1-\nu) \Gamma(\nu) {}_2F_1(\nu, 1-\nu; 1; \epsilon_2 y^2), \tag{B.1}$$

$$\int_{\epsilon_2/y}^y dx [x(1/y - \epsilon_2 x)]^{-\nu} (x-y)^{-(1-\nu)} = -y^\nu \Gamma(1-\nu) \Gamma(\nu) {}_2F_1(\nu, 1-\nu; 1; 1 - \epsilon_2 y^2), \tag{B.2}$$

$$\int_0^y dx [x(1/y - \epsilon_2 x)]^{-\nu} (y-x)^\nu = y^{1+\nu} \Gamma(1-\nu) \Gamma(1+\nu) {}_2F_1(\nu, 1-\nu; 2; \epsilon_2 y^2), \tag{B.3}$$

$$\int_{\epsilon_2/y}^y dx [x(1/y - \epsilon_2 x)]^{-\nu} (x-y)^\nu = (y - \epsilon_2/y) y^\nu \Gamma(1-\nu) \Gamma(1+\nu) {}_2F_1(\nu, 1-\nu; 2; 1 - \epsilon_2 y^2), \tag{B.4}$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ for $z < 1$ is given in (A.1). Another useful relation is

$${}_2\tilde{F}_1(\alpha, \beta; \gamma + 1; z) = \frac{\gamma}{(\gamma - \alpha)(\gamma - \beta)} (1-z)^{\gamma-\alpha-\beta+1} \partial_z {}_2\tilde{F}_1(\gamma - \alpha, \gamma - \beta; \gamma; z). \tag{B.5}$$

Note that for $\gamma - \alpha - \beta \in \mathcal{Z}$, there is no branch ambiguity in (B.5).

References

- [1] D. Lüüst, P. Mayr, R. Richter and S. Stieberger, *Scattering of gauge, matter and moduli fields from intersecting branes*, *Nucl. Phys.* **B 696** (2004) 205 [[hep-th/0404134](#)].
- [2] M. Berg, M. Haack and B. Körs, *String loop corrections to Kähler potentials in orientifolds*, *JHEP* **11** (2005) 030 [[hep-th/0508043](#)].
- [3] M. Bertolini, M. Billó, A. Lerda, J.F. Morales and R. Russo, *Brane world effective actions for D-branes with fluxes*, *Nucl. Phys.* **B 743** (2006) 1 [[hep-th/0512067](#)].
- [4] L.J. Dixon, D. Friedan, E.J. Martinec and S.H. Shenker, *The conformal field theory of orbifolds*, *Nucl. Phys.* **B 282** (1987) 13.
- [5] C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Y.S. Stanev, *Chiral asymmetry in four-dimensional open-string vacua*, *Phys. Lett.* **B 385** (1996) 96 [[hep-th/9606169](#)].
- [6] N. Akerblom, R. Blumenhagen, D. Lüüst and M. Schmidt-Sommerfeld, *Instantons and holomorphic couplings in intersecting D-brane models*, [arXiv:0705.2366](#).
- [7] E. Poppitz, *On the one loop Fayet-Iliopoulos term in chiral four dimensional type-I orbifolds*, *Nucl. Phys.* **B 542** (1999) 31 [[hep-th/9810010](#)].
- [8] L.J. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on orbifolds*, *Nucl. Phys.* **B 261** (1985) 678.
- [9] A. Strominger, *Topology of superstring compactification*, NSF-ITP-85-109, Presented at Santa Barbara Workshop on Unified String Theories, Santa Barbara, CA, Jul 29 - Aug 16, 1985.
- [10] A. Lawrence and A. Sever, *Quantum corrections to Fayet-Iliopoulos D-terms in field theory and string theory*, to appear.
- [11] G. Dvali, R. Kallosh and A. Van Proeyen, *D-term strings*, *JHEP* **01** (2004) 035 [[hep-th/0312005](#)].
- [12] P. Binetruy, G. Dvali, R. Kallosh and A. Van Proeyen, *Fayet-Iliopoulos terms in supergravity and cosmology*, *Class. and Quant. Grav.* **21** (2004) 3137 [[hep-th/0402046](#)].
- [13] P. Svrček and E. Witten, *Axions in string theory*, *JHEP* **06** (2006) 051 [[hep-th/0605206](#)].
- [14] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, [hep-th/9603167](#).
- [15] A. Lawrence and J. McGreevy, *D-terms and D-strings in open string models*, *JHEP* **10** (2004) 056 [[hep-th/0409284](#)].
- [16] D. Friedan, E.J. Martinec and S.H. Shenker, *Conformal invariance, supersymmetry and string theory*, *Nucl. Phys.* **B 271** (1986) 93.
- [17] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*, Cambridge University Press, Cambridge U.K. (1998); *String theory. Vol. 2: Superstring theory and beyond*, Cambridge University Press, Cambridge U.K. (1998).
- [18] M. Dine and M. Graesser, *CPT and other symmetries in string/M theory*, *JHEP* **01** (2005) 038 [[hep-th/0409209](#)].
- [19] E.G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and D-manifolds*, *Phys. Rev.* **D 54** (1996) 1667 [[hep-th/9601038](#)].
- [20] H. Ooguri, Y. Oz and Z. Yin, *D-branes on Calabi-Yau spaces and their mirrors*, *Nucl. Phys.* **B 477** (1996) 407 [[hep-th/9606112](#)].

- [21] J.J. Atick, L.J. Dixon and A. Sen, *String calculation of Fayet-Iliopoulos D terms in arbitrary supersymmetric compactifications*, *Nucl. Phys. B* **292** (1987) 109.
- [22] M. Dine, I. Ichinose and N. Seiberg, *F terms and D terms in string theory*, *Nucl. Phys. B* **293** (1987) 253.
- [23] P. Bain and M. Berg, *Effective action of matter fields in four-dimensional string orientifolds*, *JHEP* **04** (2000) 013 [[hep-th/0003185](#)].
- [24] I.R. Klebanov and E. Witten, *Proton decay in intersecting D-brane models*, *Nucl. Phys. B* **664** (2003) 3 [[hep-th/0304079](#)].
- [25] I. Antoniadis, C. Bachas and E. Dudas, *Gauge couplings in four-dimensional type-I string orbifolds*, *Nucl. Phys. B* **560** (1999) 93 [[hep-th/9906039](#)].
- [26] W. Fischler, H.P. Nilles, J. Polchinski, S. Raby and L. Susskind, *Vanishing renormalization of the D term in supersymmetric U(1) theories*, *Phys. Rev. Lett.* **47** (1981) 757.